1st Week

Analytical Methods in Engineering

1- Midterm : %40

1- final : %50

Attention : %10

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Chapter I

Introduction to Differential Equations

Definitions:

$\frac{dy}{dx}$ of $y = \phi(x) \Rightarrow \phi'(x)$

$$y = e^{0,1x^2}$$

$$\frac{dy}{dx} = 0,2x \cdot \underbrace{e^{0,1x^2}}_{y}$$

$$\left.\right\} \underbrace{\frac{dy}{dx} = 0,2x \cdot y}_{\text{diff. equ.}}$$

Classification by Type:

If diff. equ. contains only ordinary derivatives of one or more functions with respect to a single independent variable it is said to be ordinary diff. equ. (ODE)

An equ. involving only partial derivatives of one or more functions of two or more independent variable is called a partial diff. equ. (PDE)

ODE

$$\frac{dy}{dx} + 6y = e^{-x}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y$$

PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

ODE
Derivatives contain only one variable.

PDE
Derivatives contain more than one variables.

Classification by Order

Order of a diff. equ. is the order of the highest derivative in the equ.

$$\frac{d^{2}y}{dx^2} + 5\left(\frac{dy}{dx}\right)^{3} - 4y = e^x$$

second order, third power

$$2\frac{\partial^{4} u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

4th order

Notation:

$$y' = \frac{dy}{dx} \qquad y'' = \frac{d^2y}{dx^2} \qquad y^{(4)} = \frac{d^4y}{dx^4} \qquad \dot{y} = \frac{dy}{dt} \rightarrow \text{time}$$

Classification by Linearity

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \rightarrow \text{Linear DE}$$

- The dependent variable y and all its derivatives $y, y', y'', \ldots, y^{(n)}$ are of the first degree.
- The coefficients $a_0, a_1, \ldots, a_n$ depend at most on the independent variables

$$\left.\begin{array}{l} a_2(y) \\ (y')^2 \\ \sin(y) \\ e^y \end{array}\right\} \text{non-linear}$$

$$\left.\begin{array}{l} y'' - 2y' + y = 0 \\ x^3 \dfrac{dy^3}{dx^3} + 3x\dfrac{dy}{dx}\ 5y = e^x \end{array}\right\} \text{linear}$$

$$\left.\begin{array}{ll} (1-y)\,y' + 2y = e^x & \dfrac{d^2y}{dx^4} + y^2 = 0 \\ \dfrac{d^2y}{dx^2} + \sin y = 0 & \end{array}\right\} \text{non linear}$$

Interval Definition

open interval (a,b)

closed interval $[a,b]$

infinite interval (a,∞)

initial value problems

Solve $\dfrac{d^ny}{dx^n} = f(x, y, y', \dots y^{n-1})$

$y(x_0) = y_0 \qquad y'(x_0) = y_0 \qquad y^{(n-1)}(x_0) = y_{n-1}$

Assumptions and Hypotheses → Express assumptions in terms of D.E. → Mathematical Formulation → Solve DEs

↓

Obtain Solution → Display Predictions of the model (graph) → Check Model with known facts → If necessery alter assumptions or increase resolution

↑ (back to Assumptions and Hypotheses)

$$\frac{dT}{dt} \propto (T - T_m) \qquad \underbrace{\frac{dT}{dt} = k(T - T_m)}_{\text{cooling of a body}}$$

2nd Week

CH2. First Order Differential Equations

1-) Seperable Eqs.

When $f(x,y)$ in first order eq. $\frac{dy}{dx} = f(x,y)$ does not depend on y

$\frac{dy}{dx} = g(x)$ can be solved by integration

$$\frac{dy}{dx} = g(x) \quad dy = g(x)\,dx \Rightarrow \boxed{\int dy = \int g(x)\,dx} \rightarrow G(x)$$

$$\Rightarrow y = G(x) + c$$

If right side of the equation is seperable as

$$\frac{dy}{dx} = g(x) \cdot h(y) \Rightarrow dy = g(x) \cdot h(y) \cdot dx$$

$$\frac{dy}{h(y)} = g(x)\,dx \Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx \Rightarrow H(y) = G(x) + c$$

2-) Exact Diff. Eqs.

If the form of the solution is $f(x,y) = xy = C$

$$df = y\,dx + x\,dy = 0 \Rightarrow df = M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{df}{dx} \qquad \frac{df}{dy}$$

DE is exact when if this condition holds

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$\frac{df}{dx} = M \quad \frac{df}{dy} = N$ are used to find the function $f(x,y)$

$$\frac{df}{dx} = M(x,y) \Rightarrow df = M(x,y)\,dx \Rightarrow \int df = \int M(x,y)\,dx$$

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

Where the abritrary function $g(y)$ is the constant equation

Now differentiating equation with respect to y and assume

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

This gives

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

Ex: Solve $2xy\,dx + (x^2-1)\,dy = 0$

$M = 2xy \qquad N = x^2-1$

y'ye göre türev   x'e göre türev

$2x = 2x$ (Exact)

$M = \frac{df}{dx} = 2xy \Rightarrow df = 2xy\,dx \Rightarrow \int df = \int 2xy\,dx$

$f(x,y) = x^2y + g(y)$

$\frac{df}{dy} = x^2 + g'(y) = N = x^2 - 1$

$g'(y) = -1$

$g(y) = -y$

$f(x,y) = x^2y - y = C$

Ex: Solve the initial value problem $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$

$y(0) = 2$

$\underbrace{(\cos x \sin x - xy^2)}_{M}dx + \underbrace{y(1-x^2)}_{N}dy = 0$

$\frac{dM}{dy} = -2xy = \frac{dN}{dx}$

$$\frac{df}{dy} = y(1-x^2) \Rightarrow \int df = \int y(1-x^2)\,dy$$

$$f(x,y) = \frac{y^2}{2}(1-x^2) + h(x)$$

$$\frac{df}{dx} = -xy^2 + h'(x) = M = \cos x \sin x - xy^2$$

$$= -\frac{u^2}{2} = -\frac{1}{2}(\cos x)^2$$

$$h'(x) = \cos x \sin x \Rightarrow h(x) = -\int \overbrace{\cos x}^{u} \overbrace{(-\sin x\, dx)}^{du} = -\frac{1}{2}(\cos x)^2$$

$$\frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x = C_1 \qquad y(0) = 2$$

$$\frac{2^2}{4}(1-0^2) - \frac{1}{2}\cos^2 0 = C_1 \Rightarrow C_1 = \frac{3}{2}$$

3- Integral Multiplier Method

If equation is non exact $\frac{dM}{dy} \neq \frac{dN}{dx}$ it is sometimes possible to find an integration factor (multiplies)

$$\underbrace{P(x,y)\,M(x,y)\,dx}_{df/dx} + \underbrace{P(x,y)\,N(x,y)\,dy}_{df/dy} = 0$$

If $\frac{d[PM]}{dy} = \frac{d[PN]}{dx} \Rightarrow$ exact

$$p\frac{dM}{dy} + M\frac{dp}{dy} = p\frac{dN}{dx} + N\frac{dp}{dx}$$

$$M\frac{dp}{dy} - N\frac{dp}{dx} = p\left[\frac{dN}{dx} - \frac{dM}{dy}\right]$$

$$M\frac{1}{p}\frac{dp}{dy} - N\frac{1}{p}\frac{dp}{dx} = \frac{dN}{dx} - \frac{dM}{dy} \quad (**)$$ if $p = p(x,y)$ the

solution is difficult suppose $p = p(x)$ eq $(**)$ can be written as

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{N}\left[\frac{dM}{dy} - \frac{dN}{dx}\right] \rightarrow$$ solve as seperable

$$p(x) = e^{\int\left(\left(\frac{dM}{dy} - \frac{dN}{dx}\right)/N\right)dx}$$

if $p = p(y)$

$$\frac{1}{p}\frac{dp}{dy} = \frac{1}{M}\left[\frac{dN}{dx} - \frac{dM}{dy}\right] \Rightarrow p(y) = e^{\int\left(\left(\frac{dN}{dx} - \frac{dM}{dy}\right)/M\right)dy}$$

Ex: The nonlinear first-order diff. equ.

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0 \qquad M = xy \qquad N = 2x^2 + 3y^2 - 20$$

$$\frac{dM}{dy} = x \qquad \frac{dN}{dx} = 4x$$

*Exact yapabilmek için bir katsayı buluyoruz.

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y} \Rightarrow P(y) = e^{\int \frac{3}{y} dy} = e^{3\ln y} = y^3$$

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$$

Devamı exact problem gibi çözülecek.

$$M = xy^4 \qquad N = 2x^2y^3 + 3y^5 - 20y^3$$

$$\frac{dM}{dy} = 4xy^3 = \frac{dN}{dx} = 4xy^3$$

$$M = \frac{df}{dx} = xy^4 \Rightarrow df = xy^4\,dx \Rightarrow \int df = \int xy^4\,dx$$

$$f(x,y) = \frac{x^2}{2} y^4 + g(y)$$

$$\frac{df}{dy} = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 - 20y^3$$

$$g'(y) = 3y^5 - 20y^3$$

$$g(y) = \frac{1}{2} y^6 - 5y^4$$

$$\frac{1}{2} x^2 y^4 + \frac{1}{2} y^6 - 5y^4 = C$$

Sonuç: $\frac{1}{2} x^2 y^4 + \frac{1}{2} y^6 - 5y^4 = C$

4-) Linear ODE

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_1\frac{dy}{dx}+a_0 y=g(x)$$

if $n=1 \Rightarrow$ first order linear ODE

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \rightarrow \frac{dy}{dx}+\frac{a_0(x)}{a_1(x)}y=\frac{g(x)}{a_1(x)}$$

Stnd. form

$$\frac{dy}{dx}+p(x)y=f(x)$$

if $g(x)=0$ then ODE is called homogenous ODE

Method of soln.

$$\underbrace{\frac{dy}{dx}+P(x)y}_{\text{hom. part}}=\underbrace{f(x)}_{\text{particular part}}$$

For the homogenous part

$$\frac{dy}{dx}+P(x)y=0 \Rightarrow \frac{dy}{y}=-P(x)\,dx \Rightarrow \ln|y|=-\int P(x)dx+\ln C$$

$$y_h=Ce^{-\int P(x)dx}$$

General Solution

$$y=y_h+y_p=Ce^{-\int p(x)dx}+h(x)e^{-\int p(x)dx}$$

$y=u(x)e^{-\int p(x)dx}$ substitute $\frac{dy}{dx}+p(x)y=f(x)$

$$\frac{du}{dx}e^{-\int P dx} - Pue^{-\int P dx} + Pue^{-\int P dx} = f(x)$$

$$\frac{du}{dx}e^{-\int P dx} = f \Rightarrow du = \int e^{\int P dx} \Rightarrow u = \int fe^{\int P dx} + C$$

$$y = u \cdot e^{-\int P dx}$$

$$y = \underbrace{e^{-\int P dx}\int fe^{\int P dx}\,dx}_{y_p} + \underbrace{Ce^{-\int P dx}}_{y_h}$$

Solutions Steps

1-) $\frac{dy}{dx} + P(x)y = f(x)$ write in std form

2-) Compute $e^{\int P(x)dx}$ and multiply DE by it

$$e^{\int P dx}\frac{dy}{dx} + P(x)ye^{\int P(x)dx} = e^{\int P(x)dx} f(x)$$

3-) $\left[\frac{d}{dy}\left[y.e^{\int P(x)dx}\right] = e^{\int P(x)dx} f(x)\right]$ integrate to get $y(x)$

Ex: Solve $x\frac{dy}{dx} - 4y = x^6 e^x$

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \Rightarrow P(x) = -\frac{4}{x} \quad f(x) = x^5 e^x$$

$$e^{-4\int \frac{dx}{x}} = e^{-4\ln x} = x^{-4} \Rightarrow \frac{d}{dx}\left[x^{-4}y\right] = \underbrace{xe^x}_{f(x) \cdot e^{\int P(x)dx}}$$

$$x^{-4}y = xe^x - e^x + C$$

$$y = x^5 e^x - x^4 e^x + Cx^4$$