WEEK 3

5-) DEs with a homogeneous function

$$\frac{dy}{dx} = f(x,y)$$

$f(x,y)$ is a hom. func. if

$f(tx,ty) = t^{\alpha} f(x,y)$, α: a real number

α: describe the degree of the function

$M(x,y)dx + N(x,y)dy = 0$

is said to be homogenous if both coefficients M and N are homogenous functions of the same degree (α)

Some functions can be made homogenous

$$\frac{dy}{dx} = f(x,y) = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

we do

$$\left.\begin{aligned} x &= x_1 + h \rightarrow dx = dx_1 \\ y &= y_1 + k \rightarrow dy = dy_1 \end{aligned}\right\} \text{where } h,k \text{ are constants}$$

$$\frac{dy}{dx} = \frac{dy_1}{dx_1} = \frac{a(x_1+h)+b(y_1+k)+c}{a_1(x_1+h)+b_1(y_1+k)+c_1}$$

$$\frac{dy_1}{dx_1} = \frac{ax_1+by_1+\boxed{ah+bk+c} = 0}{a_1x_1+b_1y_1+\boxed{a_1h+b_1k+c_1} = 0}$$

$$\left.\begin{aligned} ah+bk+c &= 0 \\ a_1h+b_1k+c_1 &= 0 \end{aligned}\right\} \text{find } h,k$$

$$\frac{dy_1}{dx_1} = \frac{ax_1+by_1}{a_1x_1+b_1y_1}$$

Methods of Solution

Use either $y = u(x) \cdot x$ or $x = v(x) \cdot y$

if $y = u \cdot x \Leftrightarrow \frac{dy}{dx} = f(u)$

$$\frac{dy}{dx} = x\frac{du}{dx} + u \quad , \quad x\frac{du}{dx} + u = f(u) \Rightarrow \frac{dx}{x} = \frac{du}{f(u) - u}$$

EX: $(x^2+y^2)dx + (x^2 - xy)dy = 0$

$$\left.\begin{aligned} x^2+y^2 &\to (xt)^2 + (yt)^2 = t^2(x^2+y^2) \Rightarrow h \\ x^2 - xy &\to (xt)^2 - xt\,yt = t^2(x^2 - xy) \Rightarrow k \end{aligned}\right\} \text{homogenous } 2^{nd} \text{ degree}$$

let $y = ux$

than $dy = u\,dx + x\,du \leftarrow$ substitute in equ.

$(x^2 + u^2x^2)dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$

$x^2(1+u)\,dx + x^3(1-u)\,du = 0$

$\frac{1-u}{1+u}du + \frac{dx}{x} = 0 \quad , \quad \left[-1 + \frac{2}{1+u}\right]du + \frac{dx}{x} = 0$

integrating

$-u + 2\ln|1+u| + \ln|x| = \ln|c|$

$u = \frac{y}{x} \Rightarrow -\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c| \quad , \quad \ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x}$

or $(x+y)^2 = cxe^{y/x}$

6-) Bernoulli's DE

$\frac{dy}{dx} + P(x)y = f(x)y^n$ for $n=0, n=1$ we know

how to solve for $n \geq 2$ multiply the equation with y^{-n} we get

$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x)$

Substitute $u = y^{1-n} \overset{\text{deriv}}{\Rightarrow} (1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)P(x)\,y^{1-n}=f(x)(1-n)$$

$$\left.\frac{du}{dx}+(1-n)p(x)u=f(x)(1-n)\right\}\text{linear}$$

Note $\frac{dy}{dx}+p(x)y=f(x)\Rightarrow\frac{d}{dx}\left[ye^{\int p(x)dx}\right]=f(x)e^{\int p(x)dx}$

$$\frac{d}{dx}\left[ue^{(1-n)\int p(x)dx}\right]=(1-n)f(x)e^{(1-n)\int p(x)dx}$$

$$\int d\left[ue^{(1-n)\int p(x)dx}\right]=\int(1-n)f(x)e^{(1-n)\int p(x)dx}\,dx$$

$$ue^{(1-n)\int p(x)dx}=\int(1-n)f(x)e^{(1-n)\int p(x)dx}\,dx$$

Substitute $u=y^{1-n}$

$$y^{1-n}e^{1-n\int p(x)dx}=\int(1-n)f(x)e^{(1-n)\int p(x)dx}\,dx$$

EX:

$$x\frac{dy}{dx}+y=x^2y^2 \quad \leftarrow 1/x$$

$$\frac{dy}{dx}+\frac{y}{x}=xy^2 \quad \leftarrow y^{-2}$$

$$y^{-2}\frac{dy}{dx}+\frac{y^{-1}}{x}=x$$

$\underbrace{u=y^{-1}}$ $\underbrace{-y^{-2}\frac{dy}{dx}=\frac{du}{dx}}$

Take derivative with respect to x

$$-\frac{du}{dx}+\frac{1}{x}u=x$$

$$\frac{du}{dx} - \frac{1}{x}u = -x, \quad \frac{d}{dx}\left[ue^{\int -\frac{1}{x}dx}\right] = -xe^{\int -\frac{1}{x}dx}$$

$$\frac{d}{dx}\left[ux^{-1}\right] = -1$$

$$ux^{-1} = -x + c$$

$$u = -x^2 + cx$$

$$\frac{1}{y} = -x^2 + cx$$

$$y = \frac{1}{-x^2 + cx}$$

7-) Reduction to Seperation of Variables

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with seperable variables by means of the substitution

$U = Ax + By + C \quad B \neq 0$

Ex: $\frac{dy}{dx} = (-2x+y)^2 + 7 \quad y(0) = 0$

let $u = -2x + y \quad \frac{du}{dx} = -2 + \frac{dy}{dx}$ and substitute in diff. equ.

$$\left.\begin{array}{l} \frac{du}{dx} + 2 = u^2 - 7 \\ \frac{du}{dx} = u^2 - 9 \end{array}\right\} \begin{array}{l} \frac{du}{u^2-9} = dx \\ \frac{du}{(u-3)(u+3)} = dx \end{array}$$

$$\int \frac{1}{6}\left[\frac{1}{u-3}+\frac{1}{u+3}\right] du = \int dx$$

$$\frac{1}{6}\ln\left(\frac{u-3}{u+3}\right) = x + C \rightarrow \frac{u-3}{u+3} = e^{6x+6c} = ce^{6x}$$

$$u = \frac{3(1+Ce^{6x})}{1-Ce^{6x}} \quad , \quad y = 2x + \frac{3(1+Ce^{6x})}{1-Ce^{6x}}$$

$y(0)=0 \qquad c=-1$

$$y = 2x + \frac{3(1-e^{6x})}{1+e^{6x}}$$

8-) Lagrange's DE

$y = M(y')x + N(y') \Rightarrow y' = u(x)$

$y = M(u)x + N(u)$

$$\frac{dy}{dx} = \frac{dM(u)x}{dx} + M(u) + \frac{dN(u)}{dx}$$

$$u(x) = \underbrace{\frac{dM(u)}{du}}_{M_u} \cdot \frac{du}{dx} x + M(u) + \underbrace{\frac{dN(u)}{du}}_{N_u} \cdot \frac{du}{dx}$$

$$u - M(u) = \frac{du}{dx}\left[M_u \cdot x + N_u\right] \quad *$$

$$\frac{dx}{du} = \frac{M_u x + N_u}{u - M(u)} \quad , \quad \frac{dx}{du} - \underbrace{\frac{M_u}{u - M(u)}}_{P(u)} x = \underbrace{\frac{N_u}{u - M(u)}}_{f(u)}$$

$$\left.\frac{dx}{du} + P(u) x = f(u)\right\} \text{linear}$$

$$\frac{d}{du}\left[u e^{\int P(u) du}\right] = f(u) e^{\int P(u) du}$$

integrate

$u = u(x)$ subst in equ.

$$y = M(u) x + N(u)$$

$$y = y(x)$$

if $u - M(u) = 0$ from $*$ $u = M(u)$

$$0 = \frac{du}{dx}(M_u x + N_u)$$

$$\frac{du}{dx} = 0 \quad , \quad M_u x + N_u = 0$$

Complete Solution of Linear Equations

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n(x) y = h(x)$$

n^{th} order non-homogenous (1) linear DE first, we will solve the associated homogeneous soln. of eq(1)

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n(x)\,y = 0 \quad (2)$$

Since the equ. (2) is linear, any linear combination of individual solutions is also a solution.

thus if n-linearly independent soln. $y_1, y_2, \ldots, y_n$ can be found the general soln. of (2) will be

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = \sum_{k=1}^{n} c_k y_k(x)$$

Then the particular solution Eq(1) $y_p(x)$ is obtained.

Therefore the complete soln. of linear ODE is to be

$$y(x) = y_h(x) + y_p(x)$$

-Linear equations with constant coefficients

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = h(x) \quad (1)$$

where $a_1, a_2, \ldots, a_n$ are constant

Assume $y = e^{rx}$ (2) subst (2) in (1)

$$\underbrace{(r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n)}e^{rx} = 0$$

characteristic polynomial

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

then e^{rx} is the solution for the homogenous part.

* if then roots are distinct, general solution will be

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

if one or more roots are repeated.

$r = r_1$

$$y = (c_1 + c_2 x)e^{r_1 x} + c_3 e^{r_2 x} + \cdots$$

Ex: $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 2y = 0 \quad , y = e^{rx}$

$r^3 - 2r^2 - r + 2 = 0 \qquad r_1 = 1 \qquad r_2 = -1 \qquad r_3 = 2$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}$$

* if roots are imaginery, then these roots must be in Conjugete pairs

if $r_1 = a + bi \Rightarrow r_2 = a - bi$

$$y = Ae^{(a+ib)x} + Be^{(a-ib)y}$$

$$y = e^{ax}(\overset{\to A+B}{c_1} \cos bx + \overset{\to i(A-B)}{c_2} \sin bx) \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

* if $a \mp bi$ is a double root

$$y = e^{ax}\left[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx\right]$$

* Particular Solution (Constant coefficients)

This method may be used if $h(x)$ consist of

x^n (m in tojen), $\sin qx$, $\cos qx$

e^{px} or products of these functions

Define families as follows

$h(x)$	family
x^m	$x^m, x^{m-1}, \ldots, x^2, x, 1$
$\sin qx$	$\sin qx, \cos qx$
$\cos qx$	" "
e^x	e^x
$x\sin qx$	$x\sin qx, x\cos qx, 1\cdot\sin qx, 1\cdot\cos qx$

EX: $y''' - y' = 2x + 1 - 4\cos x + 2e^x$

$r^3 - r = 0$ $\quad r_1 = 0$

$r(r^2 - 1) = 0$ $\quad r_2 = 1$ $\quad r_3 = -1$

$$y_h = c_1 e^{0x} + c_2 e^x + c_3 e^{-x} = c_1 + c_2 e^x + c_3 e^{-x}$$

$h(x)$	families
$2x$	$x, 1$ / x^2, x
1	$\cancel{x}$
$\cos x$	$\sin x, \cos x$
e^x	e^x / xe^x

$y_p = Ax^2 + Bx + C\cos x + D\sin x + Exe^x$

Subst. y_p in D.E.

$y_p' = 2Ax + B - C\sin x + D\cos x + E(e^x + xe^x)$

$y_p''' = C\sin x - D\cos x + E(3e^x + xe^x)$

$-2Ax - B + 2C\sin x - 2D\cos x + E(2e^x) = 2x + 1 - 4\cos x + 2e^x$

$A = -1 \quad B = -1 \quad C = 0 \quad D = 2 \quad E = 1$

$y = y_h + y_p$

Ex: $y'' + 2y' + 2y = e^x \sin x$

$r^2 + 2r + 2 = 0$

$(r+1)^2 = -1$