

ES-5  $\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = A$   $0 < x < L$   $0 < t$  given  $\frac{\partial U}{\partial x}(0,t) = 0$   $U(x,0) = bx$   
 $\frac{\partial U}{\partial x}(L,t) = 0$

$U(x,t) = \psi(x,t) + w(x,t)$

$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial t} = A$

$\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial t} = 0$

$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t} = A$

$\frac{\partial U}{\partial x}(0,t) = 0$

$\frac{\partial w}{\partial x}(0,t) = 0$

$\frac{\partial \psi}{\partial x}(0,t) = 0$

$\frac{\partial U}{\partial x}(L,t) = 0$

$\frac{\partial w}{\partial x}(L,t) = 0$

$\frac{\partial \psi}{\partial x}(L,t) = 0$

$U(x,0) = bx$

$w(x,0) = bx$

$\psi(x,0) = 0$

I  $w(x,t) = X(x) \cdot T(t) \Rightarrow X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x \Rightarrow 0 \Rightarrow C_1 = 0$

$T(t) = C_3 e^{-\lambda^2 t} \Rightarrow X(x) = C_2 \cos \frac{n\pi x}{L}$

$0 \Rightarrow \lambda = \frac{n\pi}{L}$

$T(t) = C_3 e^{-(\frac{n\pi}{L})^2 t}$

$w(x,t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} e^{-(\frac{n\pi}{L})^2 t}$ ,  $w(x,0) = bx = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L}$

$C_n = \frac{2}{L} \int_0^L bx \cos \frac{n\pi x}{L} dx$ ,  $C_0 = \frac{2}{L} \int_0^L bx dx = bL$ ,  $C_n = \frac{2bL}{n^2 \pi^2} (\cos n\pi - 1)$

II  $\psi(x,t) = \sum_{n=1}^{\infty} F(t) \sin \frac{n\pi x}{L}$

$\sum_{n=1}^{\infty} -F \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} - \frac{dF}{dt} \sin \frac{n\pi x}{L} = A = \sum_{n=1}^{\infty} \underbrace{\left(-\frac{dF}{dt} - F \left(\frac{n\pi}{L}\right)^2\right)}_{d_n(t)} \sin \frac{n\pi x}{L} = A$

$d_n(t) = \frac{2}{L} \int_0^L A \sin \frac{n\pi x}{L} dx = -\frac{2}{L} \cdot A \cdot \frac{L}{n\pi} \cos \frac{n\pi x}{L} \Big|_0^L = -\frac{2A}{n\pi} (\cos n\pi - 1)$

$\frac{dF}{dt} + \left(\frac{n\pi}{L}\right)^2 F = -d_n(t) = \frac{2A}{n\pi} (\cos n\pi - 1)$

$F_h = P_1 e^{-\left(\frac{n\pi}{L}\right)^2 t}$ ,  $F_p = P_2 \Rightarrow \left(\frac{n\pi}{L}\right)^2 \cdot P_2 = d_n \Rightarrow P_2 = \frac{1}{\left(\frac{n\pi}{L}\right)^2} d_n$

$F(0) = 0 \Rightarrow P_1 e^{-\left(\frac{n\pi}{L}\right)^2 \cdot 0} + P_2 = 0 \Rightarrow P_1 = -P_2 = -\frac{d_n}{\left(\frac{n\pi}{L}\right)^2}$

$F(t) = +\frac{d_n}{\left(\frac{n\pi}{L}\right)^2} \left(1 - e^{-\left(\frac{n\pi}{L}\right)^2 t}\right)$   
 $\psi(x,t) = \sum_{n=1}^{\infty} F(t) \sin \frac{n\pi x}{L}$

9

E S I  $\frac{\partial^2 U}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 U}{\partial t^2}$  given  $U(0,t) = 0$   $U(x,0) = 0$   
 $U(L,t) = 1$   $\frac{\partial}{\partial t} U(x,0) = 0$

$U(x,t) = V(x) W(x,t)$

$\frac{\partial^2 V}{\partial x^2} = 0$	$\frac{\partial^2 W}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 W}{\partial t^2}$
$V(0) = 0$	$W(0,t) = 0$
$V(L) = 1$	$W(L,t) = 0$
	$W(x,0) = -V(x) = -\frac{x}{L}$
	$\frac{\partial W}{\partial t}(x,0) = 0$

I  $r^2 = 0 \Rightarrow V = Ax + B$   $V(0) = B = 0$   $V(L) = A \cdot L = 1 \Rightarrow A = \frac{1}{L}$   $V(x) = \frac{x}{L}$

II  $W(x,t) = X(x) T(t)$   $X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x$   
 $\frac{1}{x} \frac{d^2 x}{dx^2} = \frac{1}{a^2 T} \frac{d^2 T}{dt^2} = -\lambda^2$   $T(t) = C_3 \sin \lambda a t + C_4 \cos \lambda a t$

①  $\rightarrow x(0) = 0 \Rightarrow C_2 = 0$   $X(x) = C_1 \sin \lambda x$

②  $\rightarrow x(L) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$   $X(x) = C_1 \sin \frac{n\pi}{L} x$

$W(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L} + D_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}$

$-\frac{x}{L} = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{L} \Rightarrow D_n = \frac{2}{L} \int_0^L \frac{x}{L} \sin \frac{n\pi x}{L} dx = \frac{2}{n\pi} \cos n\pi$

$\frac{\partial}{\partial t} W(x,0) = 0 \Rightarrow C_n = 0$

$U(x,t) = \frac{x}{L} + \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n\pi} \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}$

ES-3  $\frac{\partial^2 U}{\partial x^2} = \frac{1}{k} \frac{\partial U}{\partial t}$       given  $\frac{\partial U}{\partial x}(0,t) = U(L,t) = 0$   
 $0 < x < L$        $U(x,0) = L^2 - x^2$

$$U(x,t) = X(x) T(t) \Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{kT} \frac{dT}{dt} = -\lambda^2$$

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \Rightarrow C_2 = 0, \quad C_1 \cos \lambda L = 0 = \cos\left(\frac{(2n-1)\pi}{2}\right)$$

$$T(t) = C_3 e^{-k\lambda^2 t} \quad X(x) = C_1 \cos\left(\frac{(2n-1)\pi}{2L} x\right)$$

$$U(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n-1)\pi}{2L} x\right) e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

$$U(x,0) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n-1)\pi}{2L} x\right) = L^2 - x^2$$

$$C_0 = \frac{2}{L} \int_0^L (L^2 - x^2) dx = \frac{2}{L} \left( L^2 x - \frac{x^3}{3} \right) \Big|_0^L = \frac{2}{L} \left( L^3 - \frac{L^3}{3} \right) = \frac{4L^2}{3}$$

$$C_n = \frac{2}{L} \int_0^L (L^2 - x^2) \cos\left(\frac{(2n-1)\pi}{2L} x\right) dx = -\frac{32L^2}{(2n-1)^3 \pi^3} \cos(n\pi)$$

(14)

ES-4  $\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$   $0 < y < a$   $0 < z < b$  given  $U(0, y) = 0$   $U(z, 0) = 0$   
 $\frac{\partial}{\partial z} U(b, y) = 0$   $U(z, a) = z(b-z)$

$$U(y, z) = Z(z) \cdot Y(y)$$

$$Z(z) = C_1 \sin \lambda z + C_2 \cos \lambda z \quad Z(0) = 0 \Rightarrow C_2 = 0$$

$$Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

$$C_1 \times \cos \lambda b = 0 = \cos \frac{(2n-1)\pi}{2}$$

$$\lambda = \frac{(2n-1)\pi}{2b}$$

$$U(y, z) = C_1 \sin \left( \frac{(2n-1)\pi z}{2b} \right) \left( C_3 e^{\frac{(2n-1)\pi y}{2b}} + C_4 e^{-\frac{(2n-1)\pi y}{2b}} \right)$$

$$U(z, 0) = 0 \Rightarrow C_3 + C_4 = 0 \quad C_3 = -C_4$$

$$U(y, z) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{(2n-1)\pi z}{2b} \right) \left( e^{\frac{(2n-1)\pi y}{2b}} - e^{-\frac{(2n-1)\pi y}{2b}} \right)$$

$$U(a, z) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{(2n-1)\pi z}{2b} \right) 2 \sin h \left( \frac{(2n-1)\pi a}{2b} \right)$$

$$C_n = \frac{2}{b} \cdot \frac{1}{k} \int_0^b z(b-z) \sin \frac{(2n-1)\pi z}{2b} dz$$

$$C_n = \frac{1}{k} \left( \frac{32b^2}{(2n-1)^3 \pi^3} - \frac{8b^2 \cos(n\pi)}{(2n-1)^3 \pi^2} + \frac{16b^2 n \cos(n\pi)}{(2n-1)^3 \pi^2} \right)$$

(15)

$$ES-2 \quad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$$\text{given} \quad \frac{\partial U(x,0)}{\partial y} = 0 \quad U(0,y) = y$$

$$\frac{\partial U(x,b)}{\partial y} = 0 \quad U(a,y) = 0$$

$$U(x,y) = X(x) \cdot Y(y)$$

$$0 \rightarrow C_1 = 0$$

$$Y(y) = C_1 \sin \lambda y + C_2 \cos \lambda y$$

$$\lambda = \frac{n\pi}{b}$$

$$X(x) = C_3 e^{\lambda x} + C_4 e^{-\lambda x}$$

$$X(a) = 0 \Rightarrow C_3 e^{a\lambda} + C_4 e^{-a\lambda} = 0$$

$$C_4 = -C_3 e^{2\lambda a}$$

$$U(x,y) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi y}{b} \left[ e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}} e^{-\frac{n\pi x}{b}} \right]$$

$$U(0,y) = y = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi y}{b} \left( 1 - e^{\frac{2n\pi a}{b}} \right)$$

$$C_n = \frac{2}{b} \frac{1}{k} \int_0^b y \cdot \cos \frac{n\pi y}{b} dy = \frac{2b}{k} \frac{(-1 + (-1)^n)}{n^2 \pi^2}$$

$$C_0 = \frac{2}{bk} \int_0^b y dy = \frac{b}{k}$$

HW 5-2  $\frac{\partial^2 U}{\partial x^2} = \frac{1}{k} \frac{\partial U}{\partial t}$  ( $0 < x < L, 0 < t$ )

given  $U(0,t) = 0$

$U(L,t) = \sin t$

$U(x,0) = 0$

$U(x,t) = U_1(x,t) + U_2(x,t)$

subst in PDE

$$\frac{\partial^2 U_1}{\partial x^2} - \frac{1}{k} \frac{\partial U_1}{\partial t} = - \underbrace{\frac{\partial^2 U_2}{\partial x^2} + \frac{1}{k} \frac{\partial U_2}{\partial t}}_{H(x,t)}$$

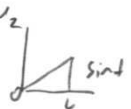
$U_1(0,t) = 0, U_2(0,t) = 0$

$U_1(L,t) = 0, U_2(L,t) = \sin t$

$U_1(x,0) = 0, U_2(x,0) = 0$

$U_1(x,t) = \sum T(t) \sin \frac{n\pi x}{L}$

$U_2(x,t) = \frac{\sin t \cdot x}{L}$



$$\Rightarrow \sum_{n=1}^{\infty} T(t) \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} - \frac{1}{k} \sum_{n=1}^{\infty} \frac{dT(t)}{dt} \sin \frac{n\pi x}{L} = \left(\frac{1}{k} \frac{\cos t}{L}\right) x$$

$$\sum_{n=1}^{\infty} \underbrace{\left[ \frac{1}{k} \frac{dT}{dt} + \left(\frac{n\pi}{L}\right)^2 T \right]}_{A_n(t)} \sin \frac{n\pi x}{L} = \frac{x \cos t}{kL}$$

$$A_n(t) = \frac{2}{L} \int_0^L \frac{x \cos t}{kL} \sin \frac{n\pi x}{L} dx = \frac{2 \cos t}{kL^2} \int_0^L x \sin \frac{n\pi x}{L} dx \Rightarrow$$

$$A_n(t) = \frac{2 \cos t}{kL^2} \left(-\frac{L^2}{n\pi} \cos n\pi\right) = \frac{1}{k} \frac{dT}{dt} + \left(\frac{n\pi}{L}\right)^2 T$$

$T = T_h + T_p \Rightarrow T_h(t) = C e^{-k \left(\frac{n\pi}{L}\right)^2 t}$

$T_p = D_1 \cos t + D_2 \sin t \Rightarrow -D_1 \sin t + D_2 \cos t + \left(\frac{n\pi}{L}\right)^2 (D_1 \cos t + D_2 \sin t) = -\frac{2 \cos n\pi}{k n\pi} (\cos t)$

$T(t) = C e^{-k \left(\frac{n\pi}{L}\right)^2 t} + D_1 \cos t + D_2 \sin t$

$U(x,0) = 0 \Rightarrow C + D_1 = 0 \Rightarrow C = -D_1$

$U_1(x,t) = \sum \left( -D_1 e^{-k \left(\frac{n\pi}{L}\right)^2 t} + D_1 \cos t + D_2 \sin t \right) \sin \frac{n\pi x}{L}$

$U(x,t) = U_1(x,t) + U_2(x,t)$

$$= \frac{\sin t \cdot x}{L} + \sum_{n=1}^{\infty} \left( -D_1 e^{-k \left(\frac{n\pi}{L}\right)^2 t} + D_1 \cos t + D_2 \sin t \right) \sin \frac{n\pi x}{L}$$



(12)

$$\text{HW 3-3} \quad \frac{\partial^2 U}{\partial x^2} = \frac{1}{k} \frac{\partial U}{\partial t}$$

$$\text{given } \frac{\partial U}{\partial x}(0,t) = 0$$

$$U(x,0) = 6 + 4 \cos \frac{3\pi x}{L}$$

$$\frac{\partial U}{\partial x}(L,t) = 0$$

$$U(x,t) = X(x)T(t)$$

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$T(t) = C_3 e^{-k + \lambda^2 t}$$

$$U(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) C_3 e^{-k + \lambda^2 t}$$

$$\frac{\partial U}{\partial x}(0,t) = 0 \Rightarrow -C_1 \lambda \sin \lambda x + C_2 \lambda \cos \lambda x = 0 \Rightarrow C_2 = 0$$

$$\frac{\partial U}{\partial x}(L,t) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

$$U(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$U(x,0) = 6 + 4 \cos \frac{3\pi x}{L} = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$$

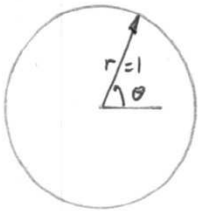
$$C_0 = \frac{2}{L} \int_0^L \left(6 + 4 \cos \frac{3\pi x}{L}\right) dx = \frac{2}{L} \left(6x + \frac{4L}{3\pi} \cos \frac{3\pi x}{L}\right) \Big|_0^L = \frac{2}{L} \left(6L + \frac{4L}{3\pi} \cdot \cos 3\pi\right) = \frac{4L}{3\pi}$$

$$\frac{C_0}{2} = 6$$

$$C_n = \frac{4 \cdot (n^2 - 27) \sin(n\pi)}{(n^3 - 9n)\pi} \Rightarrow \begin{matrix} C_{1,2,4,6,\dots} = 0 \\ C_3 = 4 \end{matrix}$$

$$U(x,t) = \left(6 + 4 \cos \frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

$$4-2 \quad \nabla^2 T = \frac{1}{r^2} \left( r^2 \frac{\partial^2 T}{\partial r^2} + r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} \right) = 0 \quad \text{given} \\ T(1, \theta) = 100 \sin 2\theta$$



$$\left. \begin{aligned} T(r, \theta) &= T(r, 2\pi) \\ \frac{\partial T}{\partial \theta}(r, \theta) &= \frac{\partial T}{\partial \theta}(r, 2\pi) \end{aligned} \right\} \text{Periodic boundary conditions.}$$

$$T(r, \theta) = R(r) \cdot \phi(\theta) \Rightarrow \frac{r^2 \frac{d^2 R}{dr^2}}{R} + \frac{r}{R} \frac{dR}{dr} = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2} = \lambda^2$$

$$R = r^m \Rightarrow R(r) = C_3 r^\lambda + C_4 r^{-\lambda}$$

$$\frac{d^2 \phi}{d\theta^2} + \lambda^2 \phi = 0 \Rightarrow \phi(\theta) = C_1 \sin \lambda \theta + C_2 \cos \lambda \theta$$

$$\left. \begin{aligned} T(r, \theta) &= T(r, 2\pi) \\ \frac{\partial T}{\partial \theta}(r, \theta) &= \frac{\partial T}{\partial \theta}(r, 2\pi) \end{aligned} \right\} \Rightarrow \lambda = n \quad \begin{array}{l} T(0, \theta) : \text{finite} \\ R(0) = \text{finite} \quad C_3 0^n - C_4 0^{-n} : \text{finite.} \\ n \geq 1 \Rightarrow C_4 = 0 \end{array}$$

$$\phi(\theta) = C_1 \cos(n\theta) + C_2 \sin(n\theta), \quad R(r) = C_3 r^n$$

$$T(r, \theta) = C_1 C_3 r^n \cos(n\theta) + C_2 C_3 r^n \sin(n\theta) \quad n = 1, 2, \dots$$

$$T(r, \theta) = \sum_{n=1}^{\infty} d_{1n} r^n \cos(n\theta) + d_{2n} r^n \sin(n\theta)$$

$$T(1, \theta) = 100 \sin 2\theta = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} 100 \sin 2\theta \cos n\theta d\theta = 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} n=2?$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} 100 \sin 2\theta \sin n\theta d\theta = 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} b_2 = 100.$$

$$T(r, \theta) = 100 r^2 \sin 2\theta$$



HW 4-3

$$4 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \quad 0 < x < 5 \\ t > 0$$

$$\text{given. } y(0, t) = y(5, t) = 0$$

$$y(x, 0) = 0$$

$$\frac{\partial y}{\partial t}(x, 0) = \begin{cases} x & 0 \leq x \leq 5/2 \\ 5-x & 5/2 \leq x \leq 5 \end{cases}$$

$$Y = X \cdot T$$

$$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x$$

$$T(t) = C_3 \sin 2\lambda t + C_4 \cos 2\lambda t$$

$$y(0, t) = 0; \quad X(0) = 0 \Rightarrow C_2 = 0 \quad y(5, t) = 0 \Rightarrow \lambda = \frac{n\pi}{5}$$

$$y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{5} \left[ C_1 C_3 \sin \frac{2\pi n t}{5} + C_1 C_4 \cos \frac{2\pi n t}{5} \right]$$

$$= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{5} \left[ d_n \sin \frac{2\pi n t}{5} + e_n \cos \frac{2\pi n t}{5} \right]$$

$$y(x, 0) = 0 = \sum_{n=1}^{\infty} e_n \sin \frac{n\pi x}{5} \Rightarrow e_n = 0$$

$$\frac{dy}{dt}(x, 0) = \begin{cases} x & 0 \leq x \leq 5/2 \\ 5-x & 5/2 \leq x \leq 5 \end{cases} = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{5} \cdot d_n \left( \frac{2n\pi}{5} \right)$$

$$d_n = \frac{1}{n\pi} \left( \int_0^{5/2} x \sin \frac{n\pi x}{5} dx + \int_{5/2}^5 (5-x) \sin \frac{n\pi x}{5} dx \right)$$

$$d_n = 50 \cos \frac{n\pi}{2}$$

$$y(x, t) = \sum_{n=1}^{\infty} 50 \cos \left( \frac{n\pi}{2} \right) \cdot \sin \frac{n\pi x}{5}$$

HW-5.1

$$\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 6z$$

given  $U(0, z) = z(b^2 - z^2)$

$U(a, z) = 0$

$U(y, 0) = 0$

$U(y, b) = 0$

$U(y, z) = \psi(z) + w(y, z)$

$$\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 w}{\partial z^2} = 6z$$

I  
 $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} = 0$

II  
 $\frac{\partial^2 \psi}{\partial z^2} = 6z$

$U(0, z) = z(b^2 - z^2) = \psi(z) + w(0, z)$

$w(0, z) = z(b^2 - z^2) - \psi(z)$

$\psi(0) = 0$

$U(a, z) = 0 = \psi(z) + w(a, z)$

$w(a, z) = -\psi(z)$

$\psi(b) = 0$

$w(y, 0) = 0$

$w(y, b) = 0$

$\psi(z) = z^3 + Az + B$

II  $\psi'' = 6z \Rightarrow \psi' = 3z^2 + Ez + C \Rightarrow \psi = z^3 + \frac{E}{2}z^2 + Cz + B$   
 $\psi(0) = 0 \Rightarrow B = 0$   
 $\psi(b) = 0 \Rightarrow b^3 + \frac{E}{2}b^2 + Cb = 0 \Rightarrow b^2(b + \frac{E}{2} + C) = 0 \Rightarrow b + \frac{E}{2} + C = 0$   
 $\psi = z^3 + Cz \Rightarrow \psi(0) = 0 \Rightarrow C = 0, \psi(b) = 0 \Rightarrow b^3 + Cb = 0 \Rightarrow C = -b^2$   
 $\psi = -(b^2 - z^2)z$

I  $\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$   
 $w(y, 0) = 0$   
 $w(y, b) = 0$   
 $w(0, z) = z(b^2 - z^2)$   
 $w(a, z) = z(b^2 - z^2)$

$w(y, z) = w_1(y, z) + w_2(y, z) \Rightarrow \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_2}{\partial z^2} + \frac{\partial^2 w_1}{\partial z^2} + \frac{\partial^2 w_2}{\partial z^2} = 0$

$\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} = 0$   
 $w_1(y, 0) = 0$   
 $w_1(y, b) = 0$   
 $w_1(0, z) = 0$   
 $w_1(a, z) = z(b^2 - z^2)$   
 $w_1(y, z) = Y_1(y) \cdot Z_1(z)$

$Z_1(z) = C_1 \sin \lambda z + C_2 \cos \lambda z \sim Z_1(0) = 0 \Rightarrow C_2 = 0, Z_1(b) = 0 \Rightarrow \lambda = \frac{n\pi}{b}$

$Y_1(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$   
 $Y_1(0) = 0 \Rightarrow C_3 + C_4 = 0 \Rightarrow C_4 = -C_3$

$Z_1(z) = C_1 \sin \frac{n\pi z}{b}$   
 $w_1(y, z) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi z}{b} \cdot 2 \sinh \frac{n\pi y}{b}$

$Y_1(y) = C_3 \left( e^{\frac{n\pi y}{b}} - e^{-\frac{n\pi y}{b}} \right)$   
 $w_1(a, z) = z(b^2 - z^2) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi z}{b} \cdot 2 \sinh \frac{n\pi a}{b}$

$C_n = \frac{1}{2 \sinh \frac{n\pi a}{b}} \int_0^a z(b^2 - z^2) \sin \frac{n\pi z}{b} dz \Rightarrow C_n = \frac{a(-ba^2 + (a-b)(a+b) \frac{n^2 \pi^2}{b^2}) \cos n\pi}{n^3 \pi^3 \cdot \sinh \frac{n\pi a}{b}}$

$w_1(y, z) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi z}{b} \sinh \frac{n\pi y}{b}$

$\frac{\partial^2 w_2}{\partial y^2} + \frac{\partial^2 w_2}{\partial z^2} = 0$   
 $w_2(y, 0) = 0$   
 $w_2(y, b) = 0$   
 $w_2(0, z) = z(z^2 - b^2)$   
 $w_2(a, z) = 0$

$w_2(y, z) = Y_2(y) Z_2(z)$

$$z_2(z) = C_5 \sin \lambda z + C_6 \cos \lambda z$$

$$z_2(0) = 0 \Rightarrow C_6 = 0$$

$$Y_2(y) = C_7 e^{\lambda y} + C_8 e^{-\lambda y}$$

$$z_2(b) = 0 \Rightarrow \lambda = \frac{n\pi}{b}$$

$$Y_2(a) = 0 \Rightarrow C_8 = -C_7 e^{2\lambda a}$$

$$z_2(z) = C_5 \sin \frac{n\pi z}{b}$$

$$Y_2(y) = C_7 e^{\frac{n\pi y}{b}} \left(1 - e^{\frac{2n\pi a}{b}}\right)$$

$$W_2(y, z) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi z}{b} e^{\frac{n\pi y}{b}} \left(1 - e^{\frac{2n\pi a}{b}}\right)$$

$$W_2(0, z) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi z}{b} \left(1 - e^{\frac{2n\pi a}{b}}\right) = z_2(z, b^2)$$

$$D_n = \frac{z}{a} \cdot \frac{1}{\left(1 - e^{\frac{2n\pi a}{b}}\right)} \int_0^a z_2(z, b^2) \sin \frac{n\pi z}{b} dz$$

$$D_n = \frac{2a(ba^2 + (b^2 - a^2)n^2\pi^2) C_5 n\pi}{n^3\pi^3}$$

$$W_2(y, z) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi z}{b} e^{\frac{n\pi y}{b}} \left(1 - e^{\frac{2n\pi a}{b}}\right)$$

$$U(y, z) = \psi(z) + W(y, z) \quad \text{where } W(y, z) = W_1(y, z) + W_2(y, z)$$

(21)

HW 5.3

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{k} \frac{\partial U}{\partial t} + e^{-x} \cdot t$$

$$0 < x < L$$

$$U(0, t) = 0$$

$$U(L, t) = 0$$

$$U(x, 0) = 0$$

$$U(x, t) = \sum_{n=1}^{\infty} F(t) \sin \frac{n\pi x}{L}$$

$$\sum_{n=1}^{\infty} -F \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} = \frac{1}{k} \sum_{n=1}^{\infty} \frac{dF}{dt} \cdot \sin \frac{n\pi x}{L} + e^{-x} \cdot t$$

$$\sum_{n=1}^{\infty} \underbrace{\left(-\left(\frac{n\pi}{L}\right)^2 F - \frac{1}{k} \frac{dF}{dt}\right)}_{a_n(t)} \sin \frac{n\pi x}{L} = e^{-x} \cdot t$$

$$a_n(t) = \frac{2}{L} \int_0^L e^{-x} \sin \frac{n\pi x}{L} dx = \frac{2+n\pi}{L^2} \cdot \frac{1 - e^{-L} \cos n\pi}{1 + \frac{n^2 \pi^2}{L^2}} \Bigg\} C_n \cdot t$$

$$-\left(\frac{n\pi}{L}\right)^2 F - \frac{1}{k} \frac{dF}{dt} = C_n t \Rightarrow \frac{dF}{dt} + \left(\frac{n\pi}{L}\right)^2 \cdot k \cdot F = C_n \cdot k \cdot t$$

$$F_h = C_1 e^{-k \left(\frac{n\pi}{L}\right)^2 t}$$

$$F_p = C_2 + t C_3 \xrightarrow{\text{substitution}} C_2 + \left(\frac{n\pi}{L}\right)^2 \cdot k \cdot (C_2 + C_3 t) = C_n k t$$

$$C_2 = C_n \left(\frac{L}{n\pi}\right)^2$$

$$C_3 = -C_n \frac{1}{k} \left(\frac{L}{n\pi}\right)^4$$

$$F = F_h + F_p = C_1 e^{-k \left(\frac{n\pi}{L}\right)^2 t} + C_n \left(\frac{L}{n\pi}\right)^2 + t C_n \frac{1}{k} \left(\frac{L}{n\pi}\right)^4$$

$$U(x, 0) = 0 \Rightarrow C_1 = -C_n \frac{1}{k} \left(\frac{L}{n\pi}\right)^4$$

$$F(t) = -\frac{C_n}{k} \left(\frac{L}{n\pi}\right)^4 \cdot e^{-k \left(\frac{n\pi}{L}\right)^2 t} + C_n \left(\frac{L}{n\pi}\right)^2 + \frac{C_n}{k} \left(\frac{L}{n\pi}\right)^4 t$$

$$U(x, t) = \sum_{n=1}^{\infty} F(t) \cdot \sin \frac{n\pi x}{L} \quad \text{where } C_n = \frac{2n\pi}{L^2} \cdot \frac{1 - e^{-L} \cos n\pi}{1 + \frac{n^2 \pi^2}{L^2}}$$

HW 7.1  $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$      $\textcircled{1} \frac{\partial U}{\partial x}(0, y) = 0$      $\textcircled{2} U(x, 0) = 0$      $0 < x < L$   
 $\textcircled{3} \frac{\partial U}{\partial x}(L, y) = 0$      $\textcircled{4} U(x, H) = f(x)$      $0 < y < H$

$$U(x, y) = X(x) Y(y)$$

$$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x$$

$$Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

$$\textcircled{1} \frac{dU}{dx}(0) = 0 \Rightarrow X'(0) = 0 \Rightarrow C_1 = 0 \quad \textcircled{2} \frac{\partial U}{\partial x}(L) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

$$X(x) = C_2 \cos n\pi x$$

$$U(x, y) = C_2 \cos \frac{n\pi x}{L} [C_3 e^{\lambda y} + C_4 e^{-\lambda y}]$$

$$U(x, 0) = 0 \Rightarrow C_2 \cos \frac{n\pi x}{L} [C_3 + C_4] = 0 \Rightarrow C_4 = -C_3$$

$$U(x, y) = \sum_{n=1}^{\infty} d_n \cos \frac{n\pi x}{L} \left( e^{\frac{n\pi}{L} y} - e^{-\frac{n\pi}{L} y} \right)$$

$$U(x, H) = f(x) = \sum_{n=1}^{\infty} d_n \cos \frac{n\pi x}{L} \cdot 2 \sinh\left(\frac{n\pi H}{L}\right)$$

$$d_n = \frac{2}{L \cdot 2 \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) dx, \quad d_n = \frac{1}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$U(x, y) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos \frac{n\pi x}{L} \cdot 2 \sinh\left(\frac{n\pi y}{L}\right)$$

$$7-4 \quad \frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2} + \sin \frac{2\pi x}{L}$$

$$\frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(L,t) = 0, \quad U(x,0) = f(x)$$

$$0 < x < L, \quad t > 0$$

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{k} \frac{\partial U}{\partial t} = -\frac{1}{k} \sin \frac{2\pi x}{L}$$

$$U(x,t) = \psi(x) + w(x,t)$$

I:

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{k} \frac{\partial w}{\partial t} = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{k} \sin \frac{2\pi x}{L}$$

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(L,t) = 0$$

$$\frac{\partial \psi}{\partial x}(0) = \frac{\partial \psi}{\partial x}(L) = 0$$

$$w(x,0) = f(x) - \psi(x)$$

$$\text{I} \quad w(x,t) = X(x) \cdot T(t)$$

$$X(x) = D \sin \lambda x + E \cos \lambda x$$

$$\Rightarrow D = 0$$

$$\frac{\partial T}{\partial t} = +k\lambda^2 T \Rightarrow T = F e^{-k\lambda^2 t} = F e^{-k t \left(\frac{n\pi}{L}\right)^2}$$

$$\lambda = \frac{n\pi}{L}$$

$$w(x,t) = E \cos \lambda x F e^{-k t \left(\frac{n\pi}{L}\right)^2}$$

$$w(x,t) = \sum_{n=1}^{\infty} G_n \cos\left(\frac{n\pi}{L} x\right) e^{-k t \left(\frac{n\pi}{L}\right)^2}$$

$$w(x,0) = f(x) - \psi(x) = \sum_{n=1}^{\infty} G_n \cos\left(\frac{n\pi}{L} x\right) \Rightarrow G_n = \frac{2}{L} \int_0^L (f(x) - \psi(x)) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$G_0 = \frac{2}{L} \int_0^L (f(x) - \psi(x)) dx$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{k} \sin \frac{2\pi x}{L} \Rightarrow \psi'' = -\frac{1}{k} \sin \frac{2\pi x}{L}$$

$$\psi(x) = -\frac{1}{k} \frac{L^2}{4\pi^2} \sin \frac{2\pi x}{L} + Bx + C$$

C is unknown.

$$\frac{\partial \psi}{\partial x}(0) = 0 = \frac{L}{2\pi} \frac{\partial \psi}{\partial x}(0) = 0 \Rightarrow B L = 0 \Rightarrow B = 0$$

$$\psi(x) = \frac{1}{k} \frac{L^2}{4\pi^2} \sin \frac{2\pi x}{L} + C$$

$$U(x,t) = \psi(x) + w(x,t) = \frac{1}{k} \frac{L^2}{4\pi^2} \sin \frac{2\pi x}{L} + C + \frac{G_0}{2} + \sum_{n=1}^{\infty} G_n \left( \cos \frac{n\pi}{L} x \right) e^{-k t \left(\frac{n\pi}{L}\right)^2}$$