ES-5 $\dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial U}{\partial t} = A \qquad 0<x<L \quad 0<t$ given $\dfrac{\partial}{\partial x} U(0,t)=0 \qquad U(x,0)=bx$

$\dfrac{\partial}{\partial x} U(L,t)=0$

$U(x,t) = \psi(x,t) + W(x,t)$

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t} + \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial t} = A$$

$\dfrac{\partial}{\partial x} U(0,t)=0$

$\dfrac{\partial U(L,t)}{\partial x}=0$

$U(x,0) = bx$

I

$$\frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial t} = 0$$

$\dfrac{\partial W}{\partial x}(0,t) = 0$

$\dfrac{\partial W(L,t)}{\partial x} = 0$

$W(x,0) = bx$

II

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t} = A$$

$\dfrac{\partial \psi}{\partial x}(0,t) = 0$

$\dfrac{\partial \psi}{\partial x}(L,t) = 0$

$\psi(x,0) = 0$

I $\quad W(x,t) = X(x) \cdot T(t) \Rightarrow X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x \quad \Rightarrow (1) \Rightarrow C_1 = 0$

$(2) \Rightarrow \lambda = \dfrac{n\pi}{L}$

$T(t) = C_3 e^{-\lambda^2 t} \Rightarrow X(x) = C_2 \cos \dfrac{n\pi x}{L}$

$T(t) = C_3 e^{-\left(\frac{n\pi}{L}\right)^2 t}$

$W(x,t) = \sum_{n=1}^{\infty} C_n \cos \dfrac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 t}, \quad W(x,0) = bx = \sum_{n=1} C_n \cos \dfrac{n\pi x}{L}$

$C_n = \dfrac{2}{L}\int_0^L bx \cos \dfrac{n\pi x}{L} dx, \quad C_0 = \dfrac{2}{L}\int_0^L bx\, dx = bL, \quad C_n = \dfrac{2bL}{n^2\pi^2}(\cos n\pi - 1)$

II $\quad \psi(x,t) = \sum_{n=1}^{\infty} F(t) \sin \dfrac{n\pi x}{L}$

$$\sum_{n=1}^{\infty} -F\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} - \frac{dF}{dt} \sin\frac{n\pi x}{L} = A = \sum_{n=1}^{\infty} \underbrace{\left(-\frac{dF}{dt} - F\left(\frac{n\pi}{L}\right)^2\right)}_{a_n(t)} \sin\frac{n\pi x}{L} = A$$

$$a_n(t) = \frac{2}{L}\int_0^L A \sin\frac{n\pi x}{L} dx = -\frac{2}{\cancel{L}} \cdot A \cdot \frac{\cancel{L}}{n\pi} \cos\frac{n\pi x}{L}\Big|_0^L = -\frac{2A}{n\pi}(\cos n\pi - 1)$$

$$\frac{dF}{dt} + \left(\frac{n\pi}{L}\right)^2 F = -a_n(t) = \underbrace{\frac{2A}{n\pi}(\cos n\pi - 1)}_{D_n}$$

$F_h = D_1 e^{-\left(\frac{n\pi}{L}\right)^2 t}, \quad F_p = D_2 \Rightarrow \left(\dfrac{n\pi}{L}\right)^2 \cdot D_2 = D_n \Rightarrow D_2 = \dfrac{1}{\left(\frac{n\pi}{L}\right)^2} D_n$

$F(0) = 0 \Rightarrow D_1 e^{-\left(\frac{n\pi}{L}\right)^2 \cdot 0} + D_2 = 0 \Rightarrow D_1 = -D_2 = -\dfrac{D_n}{\left(\frac{n\pi}{L}\right)^2}$

$F(t) = +\dfrac{D_n}{\left(\frac{n\pi}{L}\right)^2}\left(1 - e^{-\left(\frac{n\pi}{L}\right)^2 t}\right)$

$$\psi(x,t) = \sum_{n=1}^{\infty} F(t) \sin\frac{n\pi x}{L}$$

E S I $\frac{\partial^2 U}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 U}{\partial t^2}$ given $U(0,t)=0$ $\quad U(x,0)=0$

$U(L,t)=1$ $\quad \frac{\partial}{\partial t}U(x,0)=0$

$U(x,t) = V(x) + W(x,t)$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 W}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 W}{\partial t^2}$$

I

$\frac{\partial^2 V}{\partial x^2} = 0$

$V(0)=0$

$V(L)=1$

II

$\frac{\partial^2 W}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 W}{\partial t^2}$

$W(0,t)=0$

$W(L,t)=0$

$W(x,0) = -V(x) = -\frac{x}{L}$

$\frac{\partial W}{\partial t}(x,0) = 0$

I $\quad r^2=0 \Rightarrow V = Ax+B \quad V(0)=B=0$

$V(L) = A.L = 1 \Rightarrow A = \frac{1}{L} \quad V(x) = \frac{x}{L}$

II $\quad W(x,t) = X(x)\,T(t)$

$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{a^2T}\frac{d^2T}{dt^2} = -\lambda^2$

$X(x) = C_1 \sin\lambda x + C_2\cos\lambda x$

$T(t) = C_3\sin\lambda a t + C_4 \cos\lambda a t$

① $\rightarrow X(0)=0 \Rightarrow C_2 = 0 \quad X(x) = C_1\sin\lambda x$

② $\rightarrow X(L)=0 \Rightarrow \lambda = \frac{n\pi}{L} \quad X(x) = C_1\sin\frac{n\pi}{L}x$

$$W(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L} + D_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$-\frac{x}{L} = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{L} \Rightarrow D_n = \frac{2}{L}\int_0^L \frac{x}{L}\sin\frac{n\pi x}{L}dx = \frac{2}{n\pi}\cos n\pi$$

$\frac{\partial}{\partial t}W(x,0)=0 \Rightarrow C_n = 0$

$$U(x,t) = \frac{x}{L} + \sum_{n=1}^{\infty}\frac{2.(-1)^n}{n\pi}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

ES-3 $\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t}$ $\quad 0 < x < L$ $\quad 0 < t$ given $\frac{\partial}{\partial x}U(0,t) = U(L,t) = 0$

$U(x,0) = L^2 - x^2$

$U(x,t) = X(x)\,T(t) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{kT}\frac{dT}{dt} = -\lambda^2$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x \Rightarrow$ ①② $C_2 = 0.$ $\quad C_1 \cos\lambda L = 0 = \cos\left(\frac{(2n-1)\pi}{2}\right)$

$T(t) = C_3 e^{-k\lambda^2 t}$ $\qquad X(x) = C_1 \cos\left(\frac{(2n-1)\pi}{2L}x\right)$

$$U(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n-1)\pi}{2L}x\right) e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

$$U(x,0) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n-1)\pi x}{2L}\right) = L^2 - x^2$$

$$C_0 = \frac{2}{L}\int_0^L (L^2 - x^2)\,dx = \frac{2}{L}\left(L^2x - \frac{x^3}{3}\right)\Big|_0^L = \frac{2}{L}\left(L^3 - \frac{L^3}{3}\right) = \frac{4L^2}{3}$$

$$C_n = \frac{2}{L}\int_0^L (L^2 - x^2)\cos\left(\frac{2n-1}{2L}\pi x\right)dx = -\frac{32L^2}{(2n-1)^3\pi^3}\cos(n\pi)$$

ES-4 $\dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = 0$ $0<y<a$, $0<z<b$ given $U(0,y)=0$ $U(z,0)=0$

$\dfrac{\partial}{\partial z} U(b,y) = 0$ $U(z,a) = z(b-z)$

$$U(y,z) = Z(z) \cdot Y(y)$$

$$Z(z) = C_1 \sin \lambda z + C_2 \cos \lambda z \qquad Z(0)=0 \Rightarrow C_2 = 0$$

$$C_1 \times \cos \lambda b = 0 = \cos \frac{(2n-1)}{2}\pi$$

$$Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y} \qquad \lambda = \frac{2n-1}{2b}\pi$$

$$U(y,z) = C_1 \sin\left(\frac{2n-1}{2b}\pi z\right)\left(C_3 e^{\frac{2n-1}{2b}\pi y} + C_4 e^{-\frac{(2n-1)\pi}{2b} y}\right)$$

$$U(z,0) = 0 \Rightarrow C_3 + C_4 = 0 \quad C_3 = -C_4$$

$$U(y,z) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{2n-1}{2b}\pi z\right)\left(e^{\frac{(2n-1)\pi y}{2b}} - e^{-\frac{(2n-1)\pi y}{2b}}\right)$$

$$U(a,z) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{2n-1}{2b}\pi z\right) 2\sinh\frac{(2n-1)\pi a}{2b}$$

k

$$C_n = \frac{2}{b}\cdot\frac{1}{k}\int_0^b z(b-z)\sin\frac{(2n-1)\pi z}{2b}\,dz$$

$$C_n = \frac{1}{k}\left(\frac{32b^2}{(2n-1)^3\pi^3} - \frac{8b^2\cos(n\pi)}{(2n-1)^3\pi^2} + \frac{16b^2 n\cos(n\pi)}{(2n-1)^3\pi^2}\right)$$

ES-2 $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ given $\frac{\partial}{\partial y} U(x,0) = 0$ $U(0,y) = y$

$\frac{\partial}{\partial y} U(x,b) = 0$ $U(a,y) = 0$

$U(x,y) = X(x) . Y(y)$

$(1) \rightarrow C_1 = 0$

$\lambda = \frac{n\pi}{b}$

$Y(y) = C_1 \sin\lambda y + C_2 \cos\lambda y$

$X(x) = C_3 e^{\lambda x} + C_4 e^{-\lambda x}$

$X(a) = 0 \Rightarrow C_3 e^{a\lambda} + C_4 e^{-a\lambda} = 0$

$C_4 = -C_3 e^{2\lambda a}$

$$U(x,y) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi y}{b}\left[e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}} e^{-\frac{n\pi}{b}x}\right]$$

$$U(0,y) = y = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi y}{b}\underbrace{\left(1 - e^{\frac{2n\pi a}{b}}\right)}_{k}$$

$$C_n = \frac{2}{b}\frac{1}{k}\int_0^b y \cdot \cos\frac{n\pi y}{b} dy = \frac{2b}{k}\frac{(-1+(-1)^n)}{n^2\pi^2}$$

$$C_0 = \frac{2}{bk}\int_0^b y\,dy = \frac{b}{k}$$

HW 5-2 $\dfrac{\partial^2 U}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial U}{\partial t}$ $(0<x<L,\ 0<t)$ given $U(0,t)=0$

$U(L,t) = \sin t$

$U(x,0) = 0$

$U(x,t) = U_1(x,t) + U_2(x,t)$

subst in PDE

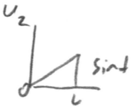
$$\frac{\partial^2 U_1}{\partial x^2} - \frac{1}{k}\frac{\partial U_1}{\partial t} = \underbrace{-\frac{\partial^2 U_2}{\partial x^2} + \frac{1}{k}\frac{\partial U_2}{\partial t}}_{H(x,t)}$$

$U_1(0,t)=0$, $U_2(0,t)=0$

$U_1(L,t)=0$ $U_2(L,t) = \sin t$

$U_1(x,0)=0$ $U_2(x,0)=0$

$U_1(x,t) = \sum T(t) \sin\frac{n\pi x}{L}$



$U_2(x,t) = \dfrac{\sin t \cdot x}{L}$

$$\Rightarrow -\sum_{n=1} T(t)\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} - \frac{1}{k}\sum_{n=1}\frac{dT(t)}{dt}\sin\frac{n\pi x}{L} = \left(\frac{1}{k}\frac{\cos t}{L}\right)x$$

$$\sum_{n=1}^{\infty}\Big[\underbrace{\frac{1}{k}\frac{dT}{dt} + \left(\frac{n\pi}{L}\right)^2 T}_{A_n(t)}\Big]\sin\frac{n\pi x}{L} = \frac{x\cos t}{kL}$$

$$A_n(t) = \frac{2}{L}\int_0^L \frac{x\cos t}{kL}\sin\frac{n\pi x}{L}\,dx = \frac{2\cos t}{kL^2}\int_0^L x\sin\frac{n\pi x}{L}dx \Rightarrow$$

$$A_n(t) = \frac{2\cos t}{kL^2}\left(-\frac{L^2}{n\pi}\cos n\pi\right) = \frac{1}{k}\frac{dT}{dt} + \left(\frac{n\pi}{L}^2\right)T$$

$T = T_h + T_p \Rightarrow$ $T_h(t) = C' e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$T_p = D_1\cos t + D_2 \sin t$, $\Rightarrow$ $-D_1\sin t + D_2\cos t + \left(\frac{n\pi}{L}\right)^2 (D_1\cos t + D_2\sin t) = -\dfrac{2\cos n\pi}{kn\pi}(\cos t)$

$$T(t) = C' e^{-k\left(\frac{n\pi}{L}\right)^2 t} + D_1\cos t + D_2\sin t$$

$U(x,0) = 0 \Rightarrow$ $C + D_1 = 0$ $C = -D_1$

$$U_1(x,t) = \sum\left(-D_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t} + D_1\cos t + D_2\sin t\right)\sin\frac{n\pi x}{L}$$

$U(x,t) = U_1(x,t) + U_2(x,t)$

$$= \frac{\sin t}{L}x + \sum_{n=1}^{\infty}\left(\left(-D_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t} + D_1\cos t + D_2\sin t\right)\sin\frac{n\pi x}{L}\right)$$

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HW 3-3 $\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t}$ given $\frac{\partial U}{\partial x}(0,t) = 0$ $\frac{\partial U}{\partial x}(L,t) = 0$ $U(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

$U(x,t) = X(x)\,T(t)$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$

$T(t) = C_3 e^{-kt\lambda^2}$

$U(x,t) = (C_1 \cos\lambda x + C_2 \sin\lambda x)\, C_3 e^{-t\lambda^2}$

$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow -C_1\lambda \sin\lambda x + C_2\lambda\cos\lambda x = 0 \Rightarrow C_2 = 0$

$\frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$

$$U(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$U(x,0) = 6 + 4\cos\frac{3\pi x}{L} = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$$

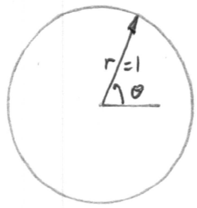
$$C_0 = \frac{2}{L}\int_0^L \left(6 + 4\cos\frac{3\pi x}{L}\right)dx = \frac{2}{L}\left(6x + \frac{4L}{3\pi}\cos\frac{3\pi x}{L}\right)\Big|_0^L = \frac{2}{L}\left[\left(6L + \frac{4L}{3\pi}\cdot\overset{+1}{\cos 3\pi}\right) - \frac{4L}{3\pi}\right.$$

$\frac{C_0}{2} = 6$

$$C_n = \frac{4\cdot(n^2-27)\sin(n\pi)}{(n^3-9n)\pi} \Rightarrow \begin{matrix} C_{1,2,4,5\ldots} = 0 \\ C_3 = 4 \end{matrix}$$

$$U(x,t) = \left(6 + 4\cos\frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

4-2 $\nabla^2 T = \frac{1}{r^2}\left(r^2\frac{\partial^2 T}{\partial^2 r} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2}\right) = 0$ given $T(1,\theta) = 100\sin 2\theta$



$$\left.\begin{array}{l} T(r,0) = T(r,2\pi) \\ \frac{\partial T}{\partial \theta}(r,0) = \frac{\partial T}{\partial \theta}(r,2\pi) \end{array}\right\}$$ Periodic boundary conditions.

$$T(r,\theta) = R(r)\cdot\phi(\theta) \Rightarrow \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda^2$$

$$R = r^m \Rightarrow R(r) = C_3 r^\lambda + C_4 r^{-\lambda}$$

$$\frac{d^2\phi}{d\theta^2} + \lambda^2\phi = 0 \Rightarrow \phi(\theta) = C_1 \sin\lambda\theta + C_2\cos\lambda\theta$$

$$\left.\begin{array}{l} T(r,0) = T(r,2\pi) \\ \frac{\partial T}{\partial \theta}(r,0) = \frac{\partial T}{\partial \theta}(r,2\pi) \end{array}\right\} \Rightarrow \lambda = n$$

$T(0,\theta)$: finite

$R(0)$ = finite $\quad C_3 0^n - C_4 0^{-n}$: finite.

$n \geq 1 \Rightarrow C_4 = 0$

$$\phi(\theta) = C_1\cos(n\theta) + C_2\sin(n\theta), \quad R(r) = C_3 r^n$$

$$T(r,\theta) = C_1C_3 r^n\cos(n\theta) + C_2C_3 r^n\sin(n\theta) \qquad n = 1,2,\ldots$$

$$T(r,\theta) = \sum_{n=1}^{\infty} d_{1n} r^n\cos(n\theta) + d_{2n} r^n \sin(n\theta)$$

$$T(1,\theta) = 100\sin 2\theta = \sum_{n=1}^{\infty} a_n\cos n\theta + b_n\sin n\theta$$

$$\left.\begin{array}{l} a_n = \frac{1}{\pi}\int_0^{2\pi} 100\sin 2\theta\cos n\theta\, d\theta = 0 \\ b_n = \frac{1}{\pi}\int_0^{2\pi} 100\sin 2\theta\sin n\theta\, d\theta = 0 \end{array}\right\}$$ $n = 2$?

$b_2 = 100$.

$$T(r,\theta) = 100 r^2 \sin 2\theta$$

HW 4-3

$$4\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \qquad 0<x<5 \qquad t>0$$

given. $y(0,t) = y(5,t) = 0$

$y(x,0) = 0$

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} x & 0 \le x \le 5/2 \\ 5-x & 5/2 \le x \le 5 \end{cases}$$

$Y = X.T$

$X(x) = C_1 \sin\lambda x + C_2 \cos\lambda x$

$T(t) = C_3 \sin 2\lambda t + C_4 \cos 2\lambda t$

$y(0,t) = 0$; $X(0) = 0 \Rightarrow C_2 = 0 \qquad y(5,t) = 0 \Rightarrow \lambda = \frac{n\pi}{5}$

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{5}\left[C_1C_3 \sin\frac{2\pi n t}{5} + C_1C_4 \cos\frac{2\pi n t}{5}\right]$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{5}\left[d_n \sin\frac{2\pi n t}{5} + e_n \cos\frac{2\pi n t}{5}\right]$$

$$y(x,0) = 0 = \sum_{n=1}^{\infty} e_n \sin\frac{n\pi x}{5} \Rightarrow e_n = 0$$

$$\frac{dy}{dt}(x,0) = \begin{cases} x & 0 \le x \le 5/2 \\ 5-x & 5/2 \le x \le 5 \end{cases} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{5} \cdot d_n\left(\frac{2n\pi}{5}\right)$$

$$d_n = \frac{1}{n\pi}\left(\int_0^{5/2} x \sin\frac{n\pi x}{5}\,dx + \int_{5/2}^{5} (5-x) \sin\frac{n\pi x}{5}\,dx\right.$$

$$d_n = 50 \cos\frac{n\pi}{2}$$

$$y(x,t) = \sum_{n=1}^{\infty} 50 \cos\left(\frac{n\pi}{2}\right) \cdot \sin\frac{n\pi x}{5}$$

HW-5.1

$$\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 6z$$

given $U(0,z) = z(b^2 - z^2)$

$U(a,z) = 0$

$U(y,0) = 0$

$U(y,b) = 0$

$U(y,z) = \Psi(z) + W(y,z)$

$$\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 W}{\partial z^2} = 6z$$

$U(0,z) = z(b^2 - z^2) = \Psi(z) + W(0,z)$

$U(a,z) = 0 = \Psi z + W(a,z)$

I

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} = 0$$

$W(0,z) = z(b^2 - z^2) - \Psi(z)$

$W(a,z) = -\Psi(z)$

$W(y,0) = 0$

$W(y,b) = 0$

II

$$\frac{\partial^2 \Psi}{\partial z^2} = 6z$$

$\Psi(0) = 0$

$\Psi(b) = 0$

$\Psi(z) = z^3 + Az + B$

II $\Psi_h = Bz + C$, $\Psi_p = Dz^3 + Ez^2 \Rightarrow 6Dz + 2E = 6z \Rightarrow D = 1$, $E = 0$

$\Psi = Bz + C + z^3 \Rightarrow \Psi(0) = 0 \Rightarrow C = 0$, $\Psi(b) = 0 \Rightarrow B(b) + b^3 = 0 \Rightarrow B = -b^2$

$\Psi = -(b^2 - z^2)z$

I $\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$ $W(y,0) = 0$ $W(y,b) = 0$ $W(0,z) = 2z(b^2 - z^2)$ $W(a,z) = z(b^2 - z^2)$

$W(y,z) = W_1(y,z) + W_2(y,z) \Rightarrow \frac{\partial^2 W_1}{\partial y^2} + \frac{\partial^2 W_2}{\partial y^2} + \frac{\partial^2 W_1}{\partial z^2} + \frac{\partial^2 W_2}{\partial z^2} = 0$

$\frac{\partial^2 W_1}{\partial y^2} + \frac{\partial^2 W_1}{\partial z^2} = 0$ $W_1(y,0) = 0$ $W_1(y,b) = 0$ $W_1(0,z) = 0$ $W_1(a,z) = z(b^2 - z^2)$ $\Big\}$ $W_1(y,z) = Y_1(y) \cdot Z_1(z)$.

$Z_1(z) = C_1 \sin\lambda z + C_2 \cos\lambda z$ $\sim$ $Z_1(0) = 0 \Rightarrow C_2 = 0$ $Z_1(b) = 0 \Rightarrow \lambda = \frac{n\pi}{b}$

$Y_1(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$

$y_1(0) = 0 \Rightarrow C_3 + C_4 = 0$ $C_4 = -C_3$

$Z_1(z) = C_1 \sin\frac{n\pi z}{b}$

$Y_1(y) = C_3\left(e^{\frac{n\pi y}{b}} - e^{-\frac{n\pi y}{b}}\right)$

$W_1(y,z) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi z}{b} \cdot 2\sinh\frac{n\pi y}{b}$

$W_1(a,z) = z(b^2 - z^2) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi z}{b} 2\sinh\frac{n\pi a}{b}$

$$C_n = \frac{1}{a\sinh\frac{n\pi a}{b}} \int_0^a z(b^2 - z^2) \sin\frac{n\pi}{a} z \, dz \Rightarrow C_n = \frac{a(-6a^2 + (a-b)(a+b) n^2\pi^2) \cos n\pi}{n^3\pi^3 \cdot \sinh\frac{n\pi a}{b}}$$

$W_1(y,z) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi z}{b} \sinh\frac{n\pi y}{b}$

$\frac{\partial^2 W_2}{\partial y^2} + \frac{\partial^2 W_2}{\partial z^2} = 0$ $W_2(y,0) = 0$ $W_2(y,b) = 0$ $W_2(0,z) = 2z(z^2 - b^2)$ $W_2(a,z) = 0$

$W_2(y,z) = Y_2(y) Z_2(t)$

$$Z_2(z) = C_5 \sin \lambda z + C_6 \cos \lambda z$$

$$Y_2(y) = C_7 e^{\lambda y} + C_8 e^{-\lambda y}$$

$$Z_2(0) = 0 \Rightarrow C_6 = 0$$

$$Z_2(b) = 0 \Rightarrow \lambda = \frac{n\pi}{b}$$

$$Y_2(a) = 0 \Rightarrow C_8 = -C_7 e^{2\lambda a}$$

$$Z_2(z) = C_5 \sin \frac{n\pi z}{b}$$

$$Y_2(y) = C_7 e^{\frac{n\pi y}{b}} \left(1 - e^{\frac{2n\pi a}{b}}\right)$$

$$w_2(y,z) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi z}{b} e^{\frac{n\pi y}{b}} \left(1 - e^{\frac{2n\pi a}{b}}\right)$$

$$w_2(0,z) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi z}{b} \left(1 - e^{\frac{2n\pi a}{b}}\right) = 2z(z^2 - b^2)$$

$$D_n = \frac{2}{a} \cdot \frac{1}{\left(1 - e^{2n\pi a/b}\right)} \int_0^a 2z(z^2 - b^2) \sin \frac{n\pi z}{b} dz$$

$$D_n = \frac{2a(6a^2 + (b^2 - a^2)n^2\pi^2)\cos n\pi}{n^3\pi^3}$$

$$w_2(y,z) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi z}{b} e^{\frac{n\pi y}{b}} \left(1 - e^{\frac{2n\pi a}{b}}\right)$$

$U(y,z) = \psi(z) + w(y,z)$ where $w(y,z) = w_1(y,z) + w_2(y,z)$

HW 5.3 $\dfrac{\partial^2 U}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial U}{\partial t} + e^{-x} . t$ $\quad 0<x<L \quad 0<t \quad U(0,t)=0 \quad U(L,t)=0 \quad U(x,0)=0$

$$U(x,t) = \sum_{n=1}^{\infty} F(t) \sin \frac{n\pi x}{L}$$

$$\sum_{n=1}^{\infty} -F\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} = \frac{1}{k}\sum_{n=1}^{\infty}\frac{dF}{dt} . \sin\frac{n\pi x}{L} + e^{-x} t$$

$$\sum_{n=1}^{\infty} \underbrace{\left(-\left(\frac{n\pi}{L}\right)^2 F - \frac{1}{k}\frac{dF}{dt}\right)}_{a_n(t)} \sin\frac{n\pi x}{L} = e^{-x} . t$$

$$a_n(t) = \frac{2}{L}\int_0^L e^{-x} t \ \sin\frac{n\pi x}{L}\, dx = \underbrace{\frac{2 t n\pi}{L^2} \cdot \frac{1 - e^{-L}\cos n\pi}{1 + \frac{n^2\pi^2}{L^2}}}_{C_n . t}$$

$$-\left(\frac{n\pi}{L}\right)^2 F - \frac{1}{k}\frac{dF}{dt} = C_n t \Rightarrow \frac{dF}{dt} + \left(\frac{n\pi}{L}\right)^2 . k . F = C_n . k . t$$

$$F_h = C_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$F_p = C_2 t + C_3 \xrightarrow{\text{substit in eq}} C_2 + \left(\frac{n\pi}{L}\right)^2 . k . (C_2 t + C_3) = C_n k t$$

$$C_2 = C_n\left(\frac{L}{n\pi}\right)^2$$

$$C_3 = -C_n \frac{1}{k}\left(\frac{L}{n\pi}\right)^4$$

$$F = F_h + F_p = C_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t} + C_n\left(\frac{L}{n\pi}\right)^2 t + C_n \frac{1}{k}\left(\frac{L}{n\pi}\right)^4$$

$$U(x,0) = 0 \Rightarrow C_1 = -C_n \frac{1}{k}\left(\frac{L}{n\pi}\right)^4$$

$$F(t) = -\frac{C_n}{k}\left(\frac{L}{n\pi}\right)^4 . e^{-k\left(\frac{n\pi}{L}\right)^2 t} + C_n\left(\frac{L}{n\pi}\right)^2 t - \frac{C_n}{k}\left(\frac{L}{n\pi}\right)^4$$

$$U(x,t) = \sum_{n=1}^{\infty} F(t) . \sin\frac{n\pi x}{L} \quad \text{where; } C_n = \frac{2n\pi}{L^2} . \frac{1 - e^{-L}\cos n\pi}{1 + \frac{n^2\pi^2}{L^2}}$$

HW 7.1 $\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}=0$ ① $\dfrac{\partial U}{\partial x}(0,y)=0$ ③ $U(x,0)=0$ $0<x<L$

② $\dfrac{\partial U}{\partial x}(L,y)=0$ ④ $U(x,H)=f(x)$ $0<y<H$

$U(x,y)=X(x)\,Y(y)$

$X(x)=C_1\sin\lambda x+C_2\cos\lambda x$

$Y(y)=C_3e^{\lambda y}+C_4e^{-\lambda y}$

① $\dfrac{dU}{dx}(0)=0 \Rightarrow X(0)=0 \Rightarrow C_1=0$ ② $\dfrac{\partial U}{\partial x}(L)=0 \Rightarrow \lambda=\dfrac{n\pi}{L}$

$X(x)=C_2\cos n\pi x$

$U(x,y)=C_2\cos\dfrac{n\pi x}{L}\left[C_3e^{\lambda y}+C_4e^{-\lambda y}\right]$

$U(x,0)=0 \Rightarrow C_2\cos\dfrac{n\pi x}{L}\left[C_3+C_4\right]=0 \Rightarrow C_4=-C_3$

$$U(x,y)=\sum_{n=1}^{\infty}d_n\cos\frac{n\pi x}{L}\left(e^{\frac{n\pi}{L}y}-e^{-\frac{n\pi}{L}y}\right)$$

$$U(x,H)=f(x)=\sum_{n=1}^{\infty}d_n\cos\frac{n\pi x}{L}\cdot 2\sinh\left(\frac{n\pi H}{L}\right)$$

$$d_0=\frac{\cancel{2}}{L\cdot\cancel{2}\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\,dx \quad , d_n=\frac{1}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

$$U(x,y)=\frac{d_0}{2}+\sum_{n=1}^{\infty}d_n\cos\frac{n\pi x}{L}\cdot 2\sinh\left(\frac{n\pi y}{L}\right)$$

7-4 $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \sin\frac{2\pi x}{L}$ $\quad \frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(L,t) = 0$, $U(x,0) = f(x)$

$0 < x < L$, $t > 0$

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{k}\frac{\partial U}{\partial t} = -\frac{1}{k}\sin\frac{2\pi x}{L}$$

$U(x,t) = \psi(x) + w(x,t)$

I

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{k}\frac{\partial w}{\partial t} = 0$$

$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(L,t) = 0$

$w(x,0) = f(x) - \psi(x)$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{k}\sin\frac{2\pi x}{L}$$

$\frac{\partial \psi(0)}{\partial x} = \frac{\partial \psi(L)}{\partial x} = 0$

I $w(x,t) = X(x).T(t)$ $\quad X(x) = D\sin\lambda x + E\cos\lambda x$ $\quad \Rightarrow D = 0$

$\lambda = \frac{n\pi}{L}$

$\frac{\partial T}{\partial t} = -k\lambda^2 T \Rightarrow T = Fe^{-k\lambda^2 t} = Fe^{-kt\left(\frac{n\pi}{L}\right)^2}$

$w(x,t) = E\cos\lambda x \; Fe^{-kt\lambda^2}$

$$w(x,t) = \sum_{n=1}^{\infty} G_n \cos\left(\frac{n\pi}{L}x\right) e^{-kt\left(\frac{n\pi}{L}\right)^2}$$

$$w(x,0) = f(x) - \psi(x) = \sum_{n=1}^{\infty} G_n\cos\left(\frac{n\pi x}{L}\right) \Rightarrow G_n = \frac{2}{L}\int_0^L (f(x)-\psi(x))\cos\left(\frac{n\pi x}{L}\right)dx$$

$$G_0 = \frac{2}{L}\int_0^L (f(x)-\psi(x))\,dx$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{1}{k}\sin\frac{2\pi x}{L} \Rightarrow \partial^2\psi = -\frac{1}{k}\sin\frac{2\pi x}{L}\,\partial x^2$$

$$\psi(x) = \frac{1}{k}\frac{L^2}{4\pi^2}\sin\frac{2\pi x}{L} + Bx + C$$

C is unknown.

$\frac{\partial\psi(0)}{\partial x} = 0 = \cancel{\frac{L}{2\pi}}\ \frac{\partial\psi(L)}{\partial x} = 0 \Rightarrow BL = 0 \Rightarrow B = 0$

$$\psi(x) = \frac{1}{k}\frac{L^2}{4\pi^2}\sin\frac{2\pi x}{L} + C$$

$$U(x,t) = \psi(x) + w(x,t) = \frac{1}{k}\frac{L^2}{4\pi^2}\sin\frac{2\pi x}{L} + C + \frac{G_0}{2} + \sum_{n=1}^{\infty} C_n\left(\cos\frac{n\pi}{L}x\right)e^{-kt\left(\frac{n\pi}{L}\right)^2}$$